M303/Specimen

Module Examination

Further Pure Mathematics

There are **two sections** in this examination.

In **Section 1** you should **submit answers to <u>all</u> 6 questions**. Each question is worth 11% of the total mark. You are advised to spend about two hours on this section.

In **Section 2** you should **submit answers to <u>2 out of the 3</u> questions**. Each question is worth 17% of the total mark.

Do not submit more than the required number of answers in Section 2. If you do answer more than 2 questions, only your first 2 answers will be marked.

Include all your working, as marks may be awarded for it. Start your answer to each question on a new page, clearly indicating the number of the question. Crossed out work will not be marked.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles $\circ$ to indicate any 'corner' points on a boundary that are not included and small closed circles $\bullet$ to indicate any 'corner' points that are included.

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Section 1

You should **attempt all questions**.
Each question is worth 11%.

Question 1

(a) Use congruences to prove that $n^5 - n$ is divisible by 5, for all $n \in \mathbb{Z}$. [2]

(b) Let $a = 2k + 1$ be an odd positive integer ($k \geq 0$). Use induction to prove that $a^{2^n} - 1$ is divisible by 2^{n+2} for all $n \geq 1$. [6]

(c) What conditions are required on an odd integer a so that $a^{2^n} - 1$ is divisible by 2^{n+3} for all $n \geq 1$? [3]

Question 2

(a) Let $G = \mathbb{Z}_{99} \times \mathbb{Z}_5$.

(i) Write down the order of G.

(ii) Write down the order of the element $(3, 1)$.

(iii) Write down an element of order 9. [3]

(b) Consider the following two groups, G and H, both of which have order 840.

$$G = \mathbb{Z}_{10} \times \mathbb{Z}_6 \times \mathbb{Z}_{14}$$
$$H = D_{21} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$$

(i) Write G as a direct product of cyclic groups of prime power order.

(ii) What is the largest $n \in \mathbb{N}$ for which H has a cyclic subgroup isomorphic to $\mathbb{Z}_n$?

(iii) Is G isomorphic to $\mathbb{Z}_{840}$? Justify your answer.

(iv) Write down, as a direct product of cyclic groups of prime power order, a non-cyclic abelian group of order 840 that is not isomorphic to G. [5]

(c) Let $(G, +)$ be a finite abelian group of order n. Let $g_1, \ldots, g_n$ be the n distinct elements of G. Show that $2(g_1 + \cdots + g_n) = 0$ where 0 is the identity element of G. [3]

Question 3

(a) List all the divisors of 110.

Hence, or otherwise, find

(i) $\tau(110)$

(ii) $\sigma(110)$

(iii) $\phi(110)$. [3]

(b) Evaluate the following Legendre symbols (note that 23 and 307 are prime). Fully justify your answer, giving references for any results used.

(i) $(2/307)$

(ii) $(23/307)$. [3]

(c) By reducing the coefficients modulo 3 (or otherwise), establish whether

$$f(x) = x^4 - 4x^3 + 4x^2 + 4x - 7$$

is irreducible over $\mathbb{Q}$, quoting any results that you use. [5]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : |a| \leq 1 \text{ and } |b| \leq 2\},\ B = \{(a, b) \in \mathbb{R}^2 : a + b < 0\}$$

and

$$C = A \cap B.$$

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Determine whether $(1, -1)$ is a closure point of C for the Euclidean metric $d^{(2)}$. [2]

(c) Write down in set notation the closure, interior and boundary of C for the Euclidean metric $d^{(2)}$. [4]

(d) Determine whether $(-1, 1)$ is a boundary point of C for the 'mixed' metric d in the plane given by $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + d_0(x_2, y_2)$, where d_0 denotes the discrete metric for $\mathbb{R}$. [2]

Question 5

Let $\alpha = \sqrt{2} - i$, where i is the imaginary unit; you may assume that $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = [\mathbb{Q}(i) : \mathbb{Q}] = 2$, $\sqrt{2} \notin \mathbb{Q}(i)$ and $i \notin \mathbb{Q}(\sqrt{2})$.

(a) Express α^3 in the form $a\sqrt{2} + bi$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a) or otherwise, show that $\sqrt{2}, i \in \mathbb{Q}(\alpha)$. [2]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, i)$. [1]

(d) Determine $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) Let $\beta = \frac{1}{2}(1 - \alpha)$. Determine $[\mathbb{Q}(\beta) : \mathbb{Q}]$ and find the minimal polynomial for β over $\mathbb{Q}$. [4]

Question 6

(a) Define a metric, d, on $X = \mathbb{R}^2 - \{(a, b) \in \mathbb{R}^2 : b = 0\}$ by

$$d(\mathbf{x}, \mathbf{y}) = \max\{2|x_1 - y_1|, 3|x_2 - y_2|\}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(i) Show that for $(a, b) \in X$ and $n \in \mathbb{N}$, $d((0, 1/n), (a, b)) \geq 3|b - 1/n|$. [2]

(ii) Hence, or otherwise, show that $(0, 1/n)$ is not a d-convergent sequence in X. [2]

(iii) Hence, or otherwise, show that X is not d-complete. [3]

(iv) Deduce that X is not d-sequentially compact. [1]

(b) Let

$$A = \{(a, 0) \in \mathbb{R}^2 : |a| \leq 1\} \cup \{(a, 1-a) \in \mathbb{R}^2 : a \in [0, 1]\} \cup \{(-a, 1-a) \in \mathbb{R}^2 : a \in [0, 1]\}.$$

Show that A is $d^{(2)}$-connected. (You may assume that line segments in the plane are $d^{(2)}$-connected.) [3]

Part 2

You should **attempt two questions**. If you attempt more, then only the scores from the first two questions will count.
Each question is worth 17%.

Question 7

(a) Let G be a group of order $676 = 2 \times 2 \times 13 \times 13$.

(i) Write down the order of the Sylow 2-subgroups.

(ii) Determine the possible number of Sylow 2-subgroups and Sylow 13-subgroups. [*5*]

(b) Let H be the cyclic group of order 676.

(i) Write down a group isomorphic to the Sylow 2-subgroup of H.

(ii) Write down an abelian group of order 676 that has a Sylow 2-subgroup that is not isomorphic to your answer to (b)(i). [*2*]

(c) Let H be the abelian group of order 676 given by $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13} \times \mathbb{Z}_{13}$. Write down a group isomorphic to the Sylow 2-subgroup of H. [*1*]

(d) Let K be an abelian group of order n where n is divisible by 13. How many Sylow 13-subgroups does K have? Justify your answer. [*2*]

(e) Let Dic_{169} be the dicyclic group of order 676 given by:

$$\mathrm{Dic}_{169} = \langle a, b | a^4 = e, a^2 = b^{169}, aba^3b = e \rangle.$$

(i) Write bab^{-1} in standard form.

(ii) Write down the underlying sets of two different Sylow 2-subgroups of Dic_{169} using standard form for the elements. [*4*]

(f) Let G be a group that has a unique element, a, with order 2. Prove that $a \in Z(G)$. [*3*]

Question 8

Let $R = \mathbb{Z}[\sqrt{-10}] = \{a + b\sqrt{-10} : a, b \in \mathbb{Z}\}$, and let

$$N(a + b\sqrt{-10}) = a^2 + 10b^2$$

be a norm on R, which you may assume is multiplicative.

(a) Show that the only elements $r \in R$ with $N(r) = 1$ are $r = \pm 1$. Hence, show that ± 1 are the only units in R. [2]

(b) Show that there are no elements of norm 2 or 7 in R, and hence show that 2 and 7 are irreducible elements. [3]

(c) Find all the elements of norm 14, and explain why they are irreducible. [3]

(d) Prove that the element $14 \in R$ can be expressed as a product of two irreducibles in two different ways. [1]

(e) Is 2 a prime in R? Justify your answer. [1]

(f) Prove that $J = \langle 7, \sqrt{-10})\rangle$ is not a principal ideal in R by assuming the contrary and completing the following steps:

(i) Assuming that $J = \langle x \rangle$ for some $x \in R$, show that x is either a unit or an associate of the element 7.

(ii) Show that if x and 7 are associates, then $7 \mid \sqrt{-10}$.

(iii) Show that if x is a unit, then $1 \in J$, and by expressing 1 as a linear combination of 7 and $\sqrt{-10}$ show that, again, $7 \mid \sqrt{-10}$.

(iv) Show that the conclusion $7 \mid \sqrt{-10}$ implies a contradiction. [7]

Question 9

(a) Let

$$A = \{f \in C[0,1] : |f(x) - f(y)| \leq (\sin|x| + \sin|y|)\sqrt{|x-y|} \text{ for each } x, y \in [0,1]\}$$

and

$$B = \{f \in C[0,1] : |f(x)| \leq 1 \text{ for } 0 \leq x \leq 1\}.$$

(i) Show that A is not pointwise bounded. [1]

(ii) Show that A is a $(d^{(1)}_{[0,1]}, d^{(1)})$-equicontinuous set of functions. [3]

(iii) Show that $A \cap B$ is $d_{\max}$-sequentially compact. (You may assume that A is a $d_{\max}$-closed set.) [3]

(b) Define a distance function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d(x, y) = |x^7 - y^7| + d_0(x, y) \text{ for } x, y \in \mathbb{R},$$

where d_0 is the discrete metric.

(i) Show that d is a metric on $\mathbb{R}$. [7]

(ii) Show that if (a_n) is a d-convergent sequence, then it is eventually constant. [3]

[END OF QUESTION PAPER]